To summarize, the following conclusions are noteworthy:

1) The azimuthal component of velocity is nonzero in contrast to nonrotating case in which $v_\Phi = 0$.

2) The rotating effect generates vorticity in that, for large distances, the flow pattern is irrotational in the nonrotating case. In the rotating case, the azimuthal component of velocity is responsible for vorticity for large R.

3) The wave-like terms $f_n(hR)$ (corresponding to a boundary layer) appearing in the solution, effective in a small distance from the surface of the sphere, and behaving like $\exp[-(1+i)(R_e/2)^{1/2}(R-1)]$ are essentially similar in nature in rotating and nonrotating cases.

4) The drag on the sphere remains unchanged (due to rotation) to the order δ, since v_x, v_r, and p remain unchanged to that order.

In a forthcoming paper, the preceding discussion will be extended to a compressible fluid in the presence of a magnetic field.

Reference

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Pressure Gradient Effects on Nonequilibrium Far Wakes

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THE purpose of this note is to present some results of recent calculations[1] pertaining to the far wake of re-entry vehicles. In these calculations, the effect of pressure gradient on a laminar-turbulent nonequilibrium mixture is assessed. The far wake problem has attracted many investigators, for example, Refs. 2–4; however, to the authors' knowledge, the quantitative effect of pressure gradient on a dissociated-ionized fluid medium undergoing nonequilibrium chemical kinetics has not appeared in the literature.

Analysis

An integral method of solution is employed similar to that of Ref. 5. The chemical model consists of 22 chemical reactions involving the 8 species O, N, O_2, N_2, NO, NO^+, e^-, O_2^-. The O_2^- species appears in an attachment reaction and a mutual neutralization reaction. The back rate constants for these reactions are taken from Ref. 6.

The Sutherland relation is used to compute the viscosity in the laminar region. In the turbulent region, the turbulent diffusivity model given by

$$(\mu)_{x\,>\,x\ \mathrm{transition}} = 0.02\ \delta_m \rho_e (u_e - u_0)$$

was used. This expression is attributed to Lees and Hromas[2] and Ting and Libby.[7] Here δ_m is a transformed wake thickness, ρ_e and u_e refer to the density and velocity at the edge of the wake, and u_0 refers to the velocity at the centerline of symmetry. The transition criteria given by Zeiberg[8] was employed in determining the transition point. The procedure given by Pallone, et al.[4] for calculating the transition point could have been used as an alternative method.

The pressure gradient in the wake was determined (with a slight modification) from the blast-wave theory of Sakurai.[9]

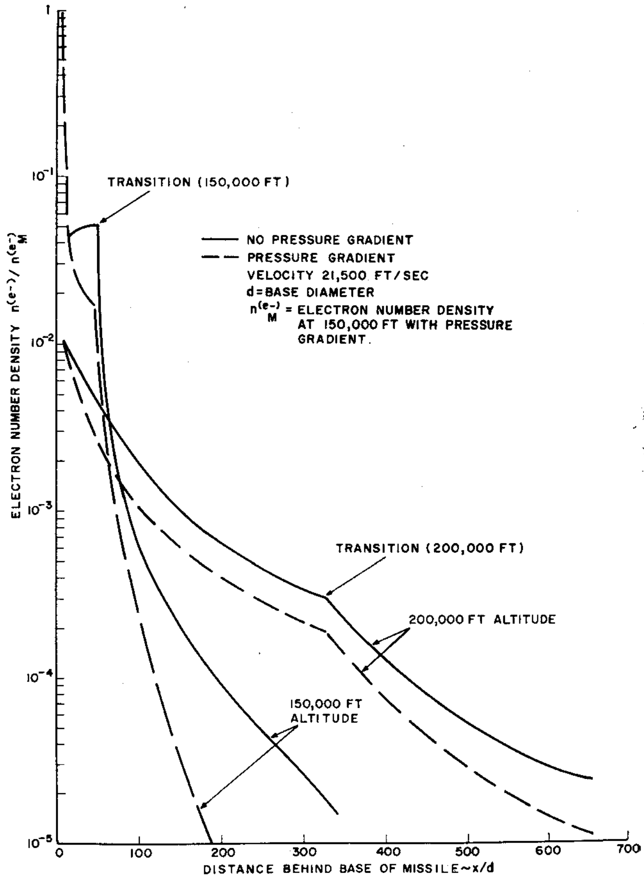

Fig. 1 Effect of pressure gradient on centerline electron density.

A similar expression has been employed by Lykoudis.[10] Because the pressure should approach ambient pressure at large distances behind the vehicle, the form of the pressure variation was taken to be

$$\frac{p}{p_\infty} = \frac{C_1 M_\infty{}^2}{x/d} + 1$$

where C_1 is a constant to be determined by theory or, preferably, by experiment. The second-order theory of Sakurai for hemisphere-cylinders yields $C_1 = 0.0665$. In the foregoing expressions, M_∞ is the freestream Mach number, x is the axial distance behind the vehicle, and d is the base diameter. For the present calculation, Sakurai's value of $C_1 = 0.0665$ was used.

The foregoing discussion presents a few of the satient features of the analysis. Complete details are contained in Ref. 1.

Results

Some results of our calculations are shown in Figs. 1 and 2. In Fig. 1, the effect of pressure gradient on the centerline electron number density is shown for two altitudes, and Fig. 2 shows the effect of pressure gradient on wake half-thickness. Because of the uncertainty of the correct initial conditions, all calculations start with equilibrium conditions.

The curves in Fig. 1 have been nondimensionalized by the same number, the electron number density at the start of the far wake at 150,000-ft alt with pressure gradient. This happens to be the maximum value for the cases shown. This method of presenting the results was used, since the intent here is to assess the effect of pressure gradient. Absolute magnitudes are not of primary concern. It can be seen that the inclusion of pressure gradient decreases the electron density over that which results from imposing a constant

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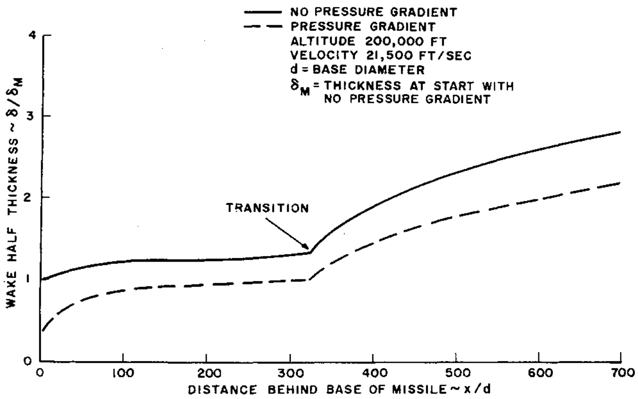

Fig. 2 Effect of pressure gradient on wake thickness.

pressure. The curves at the higher altitude have the characteristic of a partially frozen flow, whereas the lower altitude curves are characteristic of chemical equilibrium results.[11] The sharp decrease in the electron number density after transition from laminar to turbulent flow may also be noticed.

In Fig. 2, the wake half-thickness for the 200,000-ft-alt case is shown. The results for the 150,000-ft alt are qualitatively similar. The inclusion of pressure gradient decreases the wake thickness. It is noticed that the initial thickness for the pressure gradient case is less than that for constant pressure. This is a consequence of applying the momentum theorem to relate the drag of the missile to the momentum and pressure defects in the wake. It then follows (see Ref. 1 for details) from the resulting expression that an increase (over ambient) in the pressure at the starting point results in a decrease in the starting thickness.

Although numerical results are not presented here, the calculations have shown that the centerline velocity and temperature are not significantly affected by pressure gradient.

In conclusion, the results have shown that pressure gradient has a significant effect on the wake thickness and electron number density, but only a small effect on fluid velocity and temperature. Specifically, pressure gradient decreases the wake thickness and reduces the electron number density. When these two effects are combined, it may be stated that a pressure gradient produces a much smaller volume (and number) of ionized particles than does the constant pressure case.

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Technical Comments

Comment on "Derivation of Element Stiffness Matrices"

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In a recent note, Pian[1] showed that the stiffness matrix of an element can be represented by

$$k = K_{aa} - K_{ab}K_{bb}^{-1}K_{ba} \tag{1}$$

where k is an $n \times n$ matrix, the development being based upon an assumption involving $n + l$ constants, the excess l constants being adjusted to minimize the potential energy. It may be of interest that Pian's result can also be represented (in his notation) for a supported element by the simple formula

$$k = (B\,G^{-1}B^T)^{-1} \tag{2}$$

This follows from the fact that if a nonsingular matrix K and its inverse F partition into

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \text{ and } \begin{bmatrix} F_{aa} & F_{ab} \\ F_{ba} & F_{bb} \end{bmatrix} \tag{3}$$

respectively, then

$$F_{aa}^{-1} = K_{aa} - K_{ab}K_{bb}^{-1}K_{ba} \tag{4}$$

this being a well-known relation much used for eliminating coordinates from stiffness matrices.

It should be noted that Pian's matrix K inverts into $M^{-1}G^{-1}M^{-1T}$ and that

$$M^{-1} = \begin{bmatrix} B_a & B_b \\ 0 & I \end{bmatrix} \tag{5}$$

hence $F_{aa} = BG^{-1}B^T$, which gives Eq. (2) on inverting.

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